

Asymptotics of the Euler number of bipartite graphs

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Abstract

We define the Euler number of a bipartite graph on n vertices to be the number of labelings of the vertices with $1, \dots, n$ such that the vertices alternate in being local maxima and local minima. We reformulate the problem of computing the Euler number of certain subgraphs of the Cartesian product of a graph G with the path P_m in terms of self adjoint operators. The asymptotic expansion of the Euler number is given in terms of the eigenvalues of the associated operator. For two classes of graphs, the comb graphs and the Cartesian product $P_2 \square P_m$, we numerically solve the eigenvalue problem.

1 Introduction

Let $G = (V, E)$ be a bipartite graph on n vertices with the vertex decomposition $V = V_1 \cup V_2$, that is, each edge in G has one vertex in V_1 and the other in V_2 . An *alternating labeling* π is a bijection $\pi : V \rightarrow \{1, \dots, n\}$ such that for two adjacent vertices $u \in V_1$ and $v \in V_2$ we have that $\pi(u) < \pi(v)$. Another way to phrase this condition is that every vertex u in V_1 is a local minimum of the bijection π and every vertex v in V_2 is a local maximum. Define the Euler number $\mathcal{E}(G)$ to be the number of alternating labelings π of the vertices of the graph.

Two examples are in order. First, for the path P_n on n vertices the Euler number $\mathcal{E}(P_n)$ is the number of alternating permutations, that is, the classical Euler number E_n . Second, for a cycle C_n of even length n we have that $\mathcal{E}(C_n) = n/2 \cdot E_{n-1}$. Since there are $n/2$ possible positions for the largest label n , the labeling reduces to the path P_{n-1} .

Observe that we cannot drop the condition that the graph G is bipartite, since the labeling can not alternate along an odd cycle. Alternatively, for a non-bipartite graph G let $\mathcal{E}(G) = 0$. Observe that the definition of the Euler number is independent of the order of V_1 and V_2 . We also have the trivial lower bound

$$|V_1|! \cdot |V_2|! \leq \mathcal{E}(G), \quad (1.1)$$

by assigning V_2 the $|V_2|$ largest labels. Equality in (1.1) is only obtained for the complete bipartite graphs. Moreover, extending the classic ‘‘Multiplication Theorem’’ due to MacMahon [4, Article 159], for the disjoint union of two graphs G and H we have

$$\mathcal{E}(G \cup H) = \binom{m+n}{n} \cdot \mathcal{E}(G) \cdot \mathcal{E}(H), \quad (1.2)$$

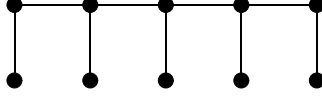


Figure 1: The comb graph Comb_m as the product $P_2 \square_{\{1\}} P_m$.

where G and H have m respectively n vertices.

Our interest is to study subgraphs of the Cartesian product of two graphs. For graphs G and H , and S a subset of vertices of G , define the product $G \square_S H$ as the graph on the vertex set $V(G) \times V(H)$ where two vertices (u, u') and (v, v') are adjacent in $G \square_S H$ if either $u = v \in S$ and u' is adjacent to v' , or u is adjacent to v and $u' = v'$. The Cartesian product $G \square H$ is obtained as a special case of the product $G \square_S H$ with $S = V(G)$.

For the general problem we obtain the following asymptotics.

Theorem 1.1. *Let G be a bipartite graph on n vertices and S a non-empty subset of the vertices of the graph G . Then there exist three positive real numbers λ , μ and c such that $\lambda > \mu$ and*

$$\frac{\mathcal{E}(G \square_S P_m)}{(m \cdot n)!} = c \cdot \lambda^{m-1} + O(\mu^{m-1}) \quad \text{as } m \longrightarrow \infty.$$

2 The self adjoint operator

For a bipartite graph $G = (V, E)$ on n vertices define two subsets X and Y of the n -dimensional unit cube $[0, 1]^V$ in n -dimensional Euclidean space \mathbb{R}^V by

$$\begin{aligned} X &= \{ \vec{x} \in [0, 1]^V : x_u + x_v \leq 1 \text{ for } \{u, v\} \in E \}, \\ Y &= \{ \vec{x} \in [0, 1]^V : x_u \leq x_v \text{ for } u \in V_1, v \in V_2, \{u, v\} \in E \}. \end{aligned}$$

Lemma 2.1. *The two subsets X and Y have the same volume, which is given by the Euler number of the graph G divided by $n!$.*

Proof. By reflecting the set Y over all of the hyperplanes of the form $x_v = 1/2$ where $v \in V_2$ we obtain the set X . Hence their volumes agree. By cutting the n -dimensional cube with the hyperplanes $x_u = x_v$ for all $u, v \in V$ we obtain $n!$ simplices of the same volume. Each simplex corresponds to a permutation by reading the order of the coordinates of a point in its interior. The set Y is the union of a subcollection of these simplices corresponding to an alternating labeling of the graph G . \square

Let the 0, 1-function χ be defined on the set $X \times X$ by

$$\chi(\vec{x}, \vec{y}) = \begin{cases} 1 & \text{if } x_v + y_v \leq 1 \text{ for all } v \in S, \\ 0 & \text{otherwise.} \end{cases}$$

Definition 2.2. Define the operator T on $L^2(X)$ by

$$T[f](\vec{x}) = \int_{\vec{y} \in X} \chi(\vec{x}, \vec{y}) \cdot f(\vec{y}) d\vec{y}.$$

Since χ is symmetric, that is, $\chi(\vec{x}, \vec{y}) = \chi(\vec{y}, \vec{x})$, the operator T is a self-adjoint Hilbert-Schmidt operator. Thus we conclude that the spectrum of T is real and discrete with 0 as the only possible accumulation point. Furthermore, all the eigenvalues and eigenfunctions of the operator are real. Since $0 \leq \chi(\vec{x}, \vec{y}) \leq 1$, the eigenvalues λ lie in the closed interval $[-1, 1]$. Hence, there is a largest eigenvalue in absolute value. Moreover, the eigenfunctions form a complete orthogonal set.

Let $\mathbf{1}$ denote the constant function with value 1 on set Y . Now we have

Proposition 2.3. For a bipartite graph G on n vertices and S a subset of the vertices of the graph G ,

$$\frac{\mathcal{E}(G \square_S P_m)}{(m \cdot n)!} = \langle \mathbf{1}, T^{m-1}[\mathbf{1}] \rangle.$$

Proof. Expanding the inner product and each of the $m - 1$ applications of the operator T , we have that

$$\begin{aligned} \langle \mathbf{1}, T^{m-1}[\mathbf{1}] \rangle &= \int_{\vec{x}_1} T^{m-1}[\mathbf{1}](\vec{x}_1) d\vec{x}_1 \\ &= \int_{\vec{x}_1} \int_{\vec{x}_2} \chi(\vec{x}_1, \vec{x}_2) \cdot T^{m-2}[\mathbf{1}](\vec{x}_2) d\vec{x}_2 d\vec{x}_1 \\ &\quad \vdots \\ &= \int_{\vec{x}_1} \int_{\vec{x}_2} \cdots \int_{\vec{x}_m} \chi(\vec{x}_1, \vec{x}_2) \cdot \chi(\vec{x}_2, \vec{x}_3) \cdots \chi(\vec{x}_{m-1}, \vec{x}_m) d\vec{x}_m \cdots d\vec{x}_2 d\vec{x}_1. \end{aligned} \quad (2.1)$$

Let \vec{x}_i be the vector $(x_{v,i})_{v \in V}$. Then the integral in equation (2.1) is over all of the $m \cdot n$ variables $x_{v,i}$ with the boundary condition that (i) $0 \leq x_{v,i} \leq 1$, (ii) $x_{v,i} + x_{v,i+1} \leq 1$ for $1 \leq i \leq m - 1$ and $v \in S$, and (iii) $x_{u,i} + x_{v,i} \leq 1$ for $\{u, v\}$ an edge in G . These inequalities describe exactly the set $X_{G \square_S P_m}$ and hence the integral is given by the ratio $\mathcal{E}(G \square_S P_m)/(m \cdot n)!$. \square

Theorem 2.4. Let G be a bipartite graph on n vertices and S a non-empty subset of the vertices of the graph G . Then we have

$$\frac{\mathcal{E}(G \square_S P_m)}{(m \cdot n)!} = \sum_{k \geq 0} \frac{\langle \varphi_k, \mathbf{1} \rangle^2}{\|\varphi_k\|^2} \cdot \lambda_k^{m-1},$$

where the eigenvalues of the operator T are $\{\lambda_k\}_{k \geq 0}$ and φ_k is the eigenfunction associated to the eigenvalue λ_k .

Proof. Expand the function $\mathbf{1}$ in terms of eigenfunctions:

$$\mathbf{1} = \sum_{k \geq 1} \frac{\langle \varphi_k, \mathbf{1} \rangle}{\|\varphi_k\|^2} \cdot \varphi_k.$$

Apply T^{m-1} and take the inner product with $\mathbf{1}$ and the result follows. \square

When the set S is empty, Theorem 2.4 is trivial. In that case $G \square_{\emptyset} P_m$ is the disjoint union of m copies of G . Using (1.2) we have that

$$\frac{\mathcal{E}(G \square_{\emptyset} P_m)}{(m \cdot n)!} = \left(\frac{\mathcal{E}(G)}{n!} \right)^m.$$

Example 2.5. When the graph G consists of a singleton vertex and S consists of this vertex, then the product $G \square_S P_m$ is exactly the path on m vertices, and its Euler number is the classical E_m . In this case the operator T is given by

$$T[f](x) = \int_0^{1-x} f(z) dz.$$

This operator has eigenvalues $\lambda_k = 2/(\pi \cdot k)$ where $k = \dots, -7, -3, 1, 5, 9, \dots$ and eigenfunctions $\varphi_k = \cos(x/\lambda_k)$. Calculating $\langle \varphi_k, \mathbf{1} \rangle = \lambda_k$ and $\|\varphi_k\|^2 = \langle \varphi_k, \varphi_k \rangle = 1/2$ we obtain the following classical asymptotic expansion for the Euler number

$$\begin{aligned} E_m &= 2 \cdot m! \cdot \sum_k \left(\frac{2}{\pi \cdot k} \right)^{m+1} \\ &= 2 \cdot m! \cdot \sum_{\substack{j \geq 1 \\ j \text{ odd}}} (-1)^{(m+1) \cdot (j-1)/2} \cdot \left(\frac{2}{\pi \cdot j} \right)^{(m+1)}, \end{aligned}$$

where $j = |k|$, that is, $k = (-1)^{(j-1)/2} \cdot j$. See [2, Section 4].

Let W be the subspace of $L^2(X)$ consisting of the functions only depending on the variables x_u where u belongs to S . In the case when S is the vertex set of the graph G the space W is $L^2(X)$. The following result applies to the case when S is strictly contained in the vertex set of G .

Proposition 2.6. *The image of the operator T is contained in subspace W . Hence all the eigenfunctions associated to non-zero eigenvalues belong to W .*

Proof. For a vertex v not in the set S , observe that the function $\chi(\vec{x}, \vec{y})$ does not depend on the variable x_v . Hence when integrating $\chi(\vec{x}, \vec{y}) \cdot f(\vec{y})$ over all $\vec{y} \in X$ the resulting function $T[f]$ does not depend on x_v , that is, $T[f]$ belongs to the space W . The second statement follows from the defining relation for eigenfunctions. \square

The Frobenius-Perron result applies to matrices, that is, linear operators on a finite-dimensional vector space. An operator version of Frobenius-Perron was discovered by Kreĭn and Rutman [3]. We present a specialized version of their result. Let Z be a measurable space. Recall that two functions in $L^2(Z)$ are considered the same if they differ on a set of measure 0. We call a function $f \in L^2(Z)$ non-negative if $f(x) \geq 0$ for almost all $x \in Z$. Similarly, we call the function f positive if $f(x) > 0$ for almost all $x \in Z$. An operator M on $L^2(Z)$ is *positivity improving* if for all non-negative but non-zero functions f the function $M[f]$ is positive.

Theorem 2.7 (Kreĭn-Rutman). *Let M be an operator $L^2(Z)$ such that there is a positive integer k so that M^k is positivity improving. Then the largest eigenvalue λ (in modulus) of M is real, positive and simple. Moreover, the associated eigenfunction φ is a positive function on Z .*

m	E_m	$\mathcal{E}(\text{Comb}_m)$	$\sim \mathcal{E}(\text{Comb}_m)$	$\mathcal{E}(P_2 \square P_m)$	$\sim \mathcal{E}(P_2 \square P_m)$
1	1	1	0.99379166	1	0.98451741
2	1	5	4.99961911	4	4.002193
3	2	66	65.9990972	44	43.99713
4	5	1613	1612.99965	896	896.0018
5	16	63480	63480.0072	29392	29391.93
6	61	3662697	3.66269757×10^6	1413792	1.413789×10^6
7	272	291407424	2.91407470×10^8	93770800	9.377064×10^7
8	1385	30572578425	$3.05725832 \times 10^{10}$	8201380224	8.201366×10^9
9	7936	4089549416832	$4.08955006 \times 10^{12}$	914570667792	9.145691×10^{11}
10	50521	679329771871725	$6.79329879 \times 10^{14}$	126651310675680	1.266511×10^{14}

Table 1: Table of the Euler numbers, the number of alternating $2 \times m$ arrays, comb graph Comb_m , and their numerical approximations, denoted by $\sim \mathcal{E}(\text{Comb}_m)$ and $\sim \mathcal{E}(P_2 \square P_m)$.

Applying Kreĭn-Rutman to our operator T , we have

Proposition 2.8. *The operator T^2 is positivity improving. The largest eigenvalue (in absolute value) λ of the operator T is real, positive and simple. Furthermore, the associated eigenfunction φ is positive.*

Proof. Let f be a non-negative, non-zero function in $L^2(X)$. By the definition of the operator T we have that in a neighborhood of 0 the function $T[f]$ has a positive support. By applying the operator T again we obtain that every point in the interior of Y takes a positive value in the function $T^2[f]$. The remainder of the proposition follows from Kreĭn-Rutman. \square

Proof of Theorem 1.1. By letting λ be the largest eigenvalue of the operator T and letting μ be a bound on the next largest eigenvalue such that $\lambda > \mu$, the result follows. \square

3 The comb graph

We now turn our attention to the comb graph. See Figure 1. Recall that the comb graph is defined by the product $P_2 \square_{\{1\}} P_m$. In this case the space X is the triangle

$$X = \{(x, y) : x, y \geq 0, x + y \leq 1\}.$$

However, following Proposition 2.6 in order to find the eigenvalue and eigenfunctions of T it is enough to consider the subspace W of $L^2(X)$ consisting of functions depending only on the variable x . Observe that W inherits the inner product

$$\langle f, g \rangle_W = \int_0^1 (1-x) \cdot f(x) \cdot \overline{g(x)} \, dx.$$

Moreover, the operator T is given by

$$T[f](x) = \int_0^{1-x} (1-z) \cdot f(z) \, dz.$$

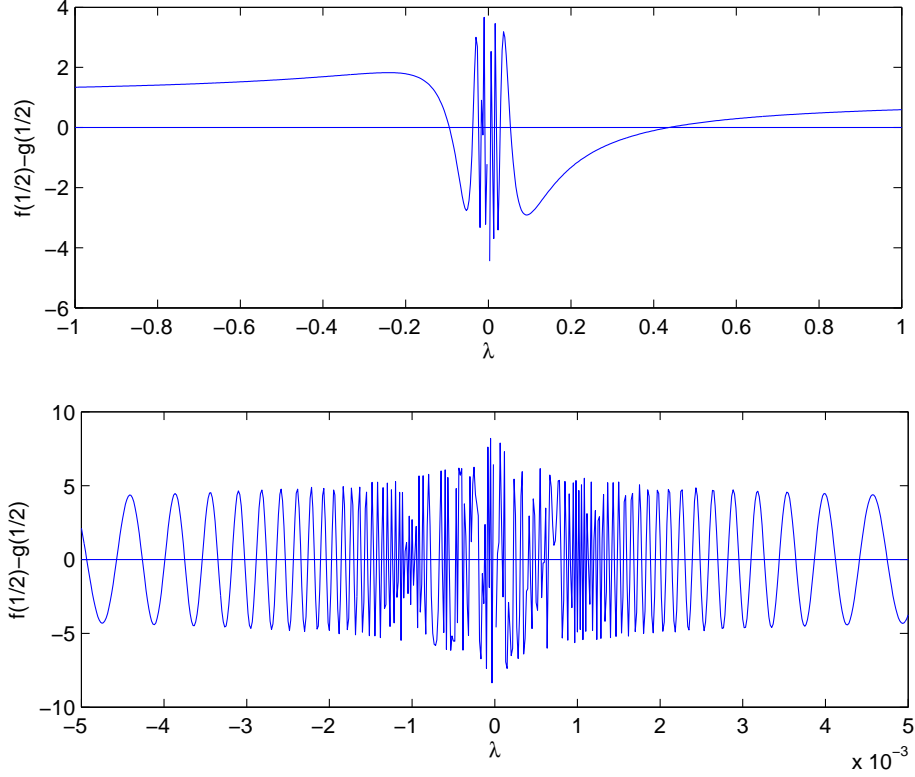


Figure 2: The difference $f(1/2) - g(1/2)$ found by solving the system of ODE in (3.3) with a given value of λ . The roots of this plot correspond to eigenvalues. The lower plot is a magnification of the center domain.

The next step is to find all of the eigenvalues and eigenfunctions of the operator T , that is, functions $f(x)$ so that

$$\lambda \cdot f(x) = T[f](x) = \int_0^{1-x} (1-z) \cdot f(z) dz. \quad (3.1)$$

We convert this integral equation into a differential equation by differentiating to obtain

$$\lambda \cdot f'(x) = -x \cdot f(1-x). \quad (3.2)$$

To convert this into an ordinary differential equation (ODE), we define $g(x) = f(1-x)$ and thus

$$\begin{pmatrix} f \\ g \end{pmatrix}' = \begin{pmatrix} 0 & -x/\lambda \\ (1-x)/\lambda & 0 \end{pmatrix} \cdot \begin{pmatrix} f \\ g \end{pmatrix}. \quad (3.3)$$

Together with the boundary conditions

$$f(0) = 1, \quad g(0) = 0, \quad f(1/2) = g(1/2), \quad (3.4)$$

which come from the integral equation (3.1) and the algebraic relationship between f and g , this linear system is equivalent to the original integral equation. The condition that $f(0) = 1$ is our choice of normalization for the eigenfunctions. The only solution which has $f(0) = g(0) = 0$ is identically zero, thus this normalization is valid. We proceed to solve this ODE numerically.

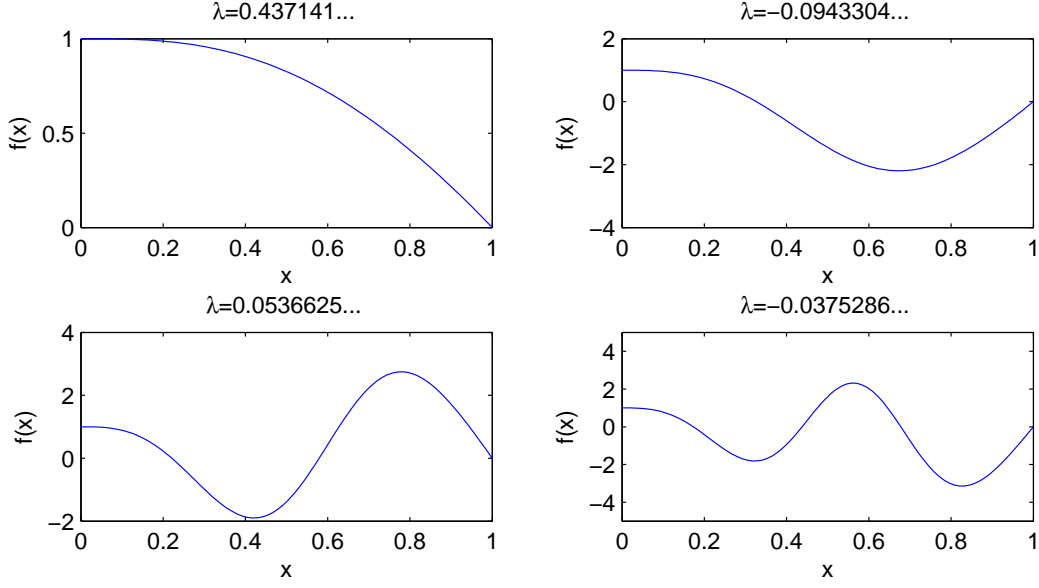


Figure 3: The eigenfunctions associated with the four largest (in absolute value) eigenvalues. The function $f(x)$, in $[0, 1]$ is found by using $f(x)$ in $[1/2, 1]$ and $g(1 - x)$ in $[0, 1/2]$.

First, we solve the system from 0 to $1/2$ for various values of λ and find the difference $f(1/2) - g(1/2)$. This is plotted in Figure 2. The roots of this plot correspond to eigenvalues and we find them numerically. For each eigenvalue λ the functions f and g are found and $f(x)$ over the unit interval is reconstructed. The eigenfunctions corresponding to the largest (in absolute value) eigenvalues are plotted in Figure 3. The resulting norms and constants c_n are tabulated in Table 2.

λ	$\langle f(x), \mathbf{1} \rangle$	$\ f(x)\ ^2$	c_n
0.437141117	0.437141151	0.398916677	0.479028320
-0.094330445	0.094331326	0.690741849	0.012882380
0.053662538	0.053688775	0.829794009	0.003473735
-0.037528586	-0.037546864	0.932757330	0.001511397

Table 2: The values of λ , $\langle f(x), \mathbf{1} \rangle_W$, $\|f(x)\|_W^2$, and c_n for the first four eigenfunctions shown in Figure 3. The constant c_n is the ratio $\langle f(x), \mathbf{1} \rangle_W^2 / \|f(x)\|_W^2$ for the n^{th} eigenfunction.

The Euler numbers for the Comb_n graphs are calculated from the numerical approximation for λ_n and c_n using the first four terms in the series in Theorem 2.4. They are tabulated in the fourth column of Table 1.

4 Alternating 2 by m arrays

The Euler number of the Cartesian product of two paths P_m and P_n counts the number of *alternating m by n arrays*. That is, the number of assignments of the integers $1, 2, \dots, m \cdot n$ to an m by n array such that each entry is a local maximum or a local minimum. Hence, if $i + j$ is even then the entry $a_{i,j}$ should be less than the four adjacent entries $a_{i-1,j}, a_{i+1,j}, a_{i,j-1}, a_{i,j+1}$. Similarly, if $i + j$ is odd then the entry $a_{i,j}$ should be larger than the four adjacent entries.

In the following we study the number of alternating 2 by m arrays, that is, the Euler number of the graph $P_2 \square P_m$. The graph G is the path on two vertices P_2 and $S = \{1, 2\}$. As before, the space X is the triangle

$$X = \{(x, y) : x, y \geq 0, x + y \leq 1\}.$$

Observe that the operator T has the form

$$T[f](x, y) = \int_R f(z, w) dz dw,$$

where R is the region described by the inequalities $0 \leq z \leq 1 - x$, $0 \leq w \leq 1 - y$ and $z + w \leq 1$. Since $x + y \leq 1$, equivalently $(1 - x) + (1 - y) \geq 1$, the inequality $z + w \leq 1$ cuts off a triangle from the rectangle $[0, 1 - x] \times [0, 1 - y]$. Hence we have

$$T[f](x, y) = \int_0^y \int_0^{1-y} f(z, w) dw dz + \int_y^{1-x} \int_0^{1-z} f(z, w) dw dz \quad (4.1)$$

$$= \int_0^x \int_0^{1-x} f(z, w) dz dw + \int_x^{1-y} \int_0^{1-w} f(z, w) dz dw. \quad (4.2)$$

In order to study this operator T , it will be easier to work in a different space. Let U be the space of functions $g(x)$ on the interval $[0, 1]$ that satisfy the inequality

$$\int_X (g(x) + g(y)) \cdot \overline{(g(x) + g(y))} dx dy < \infty.$$

Enrich the space U with the following inner product

$$\langle g, h \rangle_U = \int_X (g(x) + g(y)) \cdot \overline{(h(x) + h(y))} dx dy.$$

Define L to be the linear map $L : U \rightarrow L^2(X)$ defined by $L[g](x, y) = g(x) + g(y)$. Observe that the map L preserves the inner product, that is, L is an isometry. Moreover, L is an injective map. Furthermore, define the operator T on U by

$$T[g](x) = (1 - x) \cdot \int_0^x g(s) ds + \int_x^{1-x} (1 - s) \cdot g(s) ds. \quad (4.3)$$

The reason why we denote this operator also by T will be clear from the next proposition.

Proposition 4.1. *The isometry L and the operator T commute, that is, $T \circ L = L \circ T$.*

Proof. Apply the original operator T to the function $L[g](x, y) = g(x) + g(y)$. We do this by applying equation (4.1) to $g(x)$ and applying equation (4.2) to $g(y)$. We then have

$$\begin{aligned}
T[L[g]](x, y) &= \int_0^y \int_0^{1-y} g(z) dw dz + \int_y^{1-x} \int_0^{1-z} g(z) dw dz \\
&\quad + \int_0^x \int_0^{1-x} g(w) dz dw + \int_x^{1-y} \int_0^{1-w} g(w) dz dw \\
&= (1-y) \cdot \int_0^y g(z) dz + \int_y^{1-x} (1-z) \cdot g(z) dz \\
&\quad + (1-x) \cdot \int_0^x g(w) dw + \int_x^{1-y} (1-w) \cdot g(w) dw \\
&= L[T[g]](x, y),
\end{aligned}$$

where we used the fact that $\int_a^b + \int_c^d = \int_a^d + \int_c^b$. □

Hence we have that

$$\langle \mathbf{1}, T^{m-1}[\mathbf{1}] \rangle_{L^2(X)} = \frac{1}{4} \cdot \langle \mathbf{1}, T^{m-1}[\mathbf{1}] \rangle_U,$$

since $L[1/2 \cdot \mathbf{1}] = \mathbf{1}$. Thus it is enough to solve the eigenvalue problem $\lambda \cdot g(x) = T[g](x)$ in U for non-zero λ , that is,

$$\lambda \cdot g(x) = (1-x) \cdot \int_0^x g(s) ds + \int_x^{1-x} (1-s) \cdot g(s) ds. \quad (4.4)$$

Differentiate to obtain

$$\lambda \cdot g'(x) = - \int_0^x g(s) ds - x \cdot g(1-x). \quad (4.5)$$

Differentiate again

$$\lambda \cdot g''(x) = -g(x) - g(1-x) + x \cdot g'(1-x). \quad (4.6)$$

As before, we convert this into a linear system of differential equations by defining $h(x) = g(1-x)$.

$$\begin{pmatrix} g \\ h \\ g' \\ h' \end{pmatrix}' = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1/\lambda & -1/\lambda & 0 & -x/\lambda \\ -1/\lambda & -1/\lambda & (1-x)/\lambda & 0 \end{pmatrix} \cdot \begin{pmatrix} g \\ h \\ g' \\ h' \end{pmatrix} \quad (4.7)$$

We solve this system differential system numerically on the interval $[1/2, 1]$. To find boundary conditions we set $x = 1/2$ in equations (4.4) and (4.5).

$$\lambda \cdot g(1/2) = 1/2 \cdot \int_0^{1/2} g(s) ds, \quad (4.8)$$

$$\lambda \cdot g'(1/2) = - \int_0^{1/2} g(s) ds - 1/2 \cdot g(1/2). \quad (4.9)$$

Observe that $g(1/2) = h(1/2) = 0$, (4.8) and (4.9) imply that $g'(1/2) = h'(1/2) = 0$ and therefore

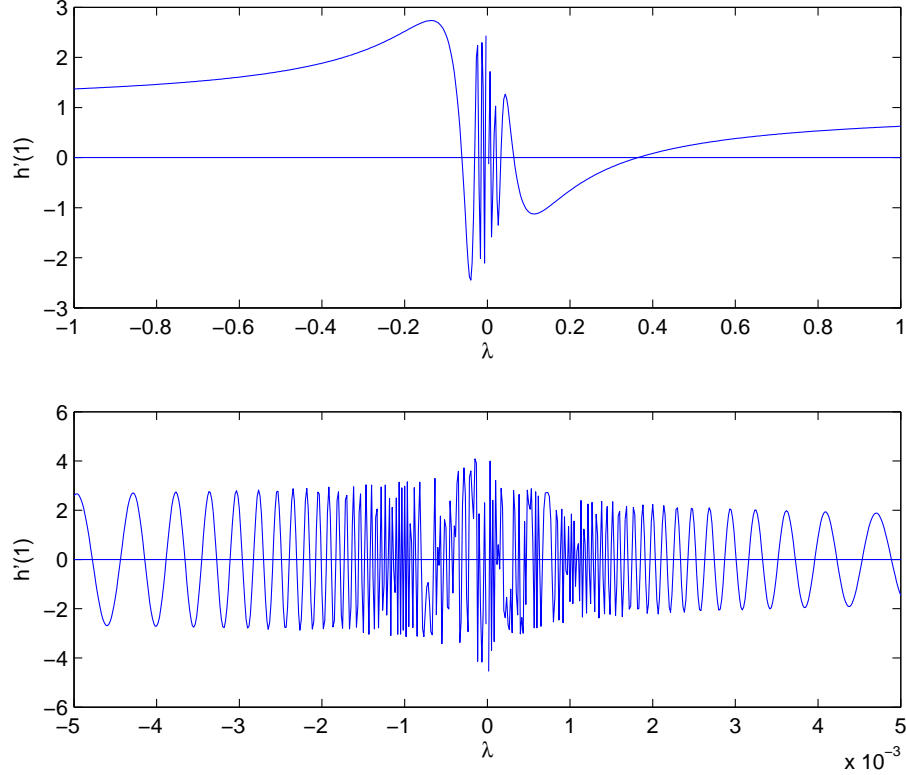


Figure 4: The value of $h'(1)$ found by solving the system of ODE's in (4.7) with a given value of λ . The roots of this plot correspond to eigenvalues. The lower plot is a magnification of the center domain.

corresponds to the zero solution of (4.7). Since we are looking for the non-zero solution, we normalize such that

$$g(1/2) = h(1/2) = \lambda/2 \neq 0, \quad (4.10)$$

This gives us two conditions at $x = 1/2$, and also implies that $\int_0^{1/2} g(x) dx = \lambda^2$. Combined with (4.9) this gives two more conditions

$$g'(1/2) = -h'(1/2) = -\lambda - \frac{1}{4}. \quad (4.11)$$

Thus given the parameter λ we can solve the system from $x = 1/2$ to $x = 1$. At $x = 1$ however, the integral equation (4.5) yields another constraint:

$$h'(1) = 0. \quad (4.12)$$

Equivalently, looking at (4.4) (and remembering that $g(x) = h(1 - x)$) for $x = 0, 1$ we find that

$$h(1) = g(0) = -g(1). \quad (4.13)$$

This is only satisfied for a discrete set of eigenvalues λ . To find this set we (numerically) solve the ODE starting from $x = 1/2$ using the initial conditions given by (4.10, 4.11) and search values of λ that give $h'(1) = 0$. Figure 4 shows the value of $h'(1)$ for various values of λ .

A numerical root-finding algorithm finds the first few roots, that is, eigenvalues λ . The associated eigenfunctions are shown in Figure 5. Finally, to find c_n we must evaluate $\langle g, \mathbf{1} \rangle_U$ and $\langle g, g \rangle_U$. This is again done numerically. The results are shown in Table 3.

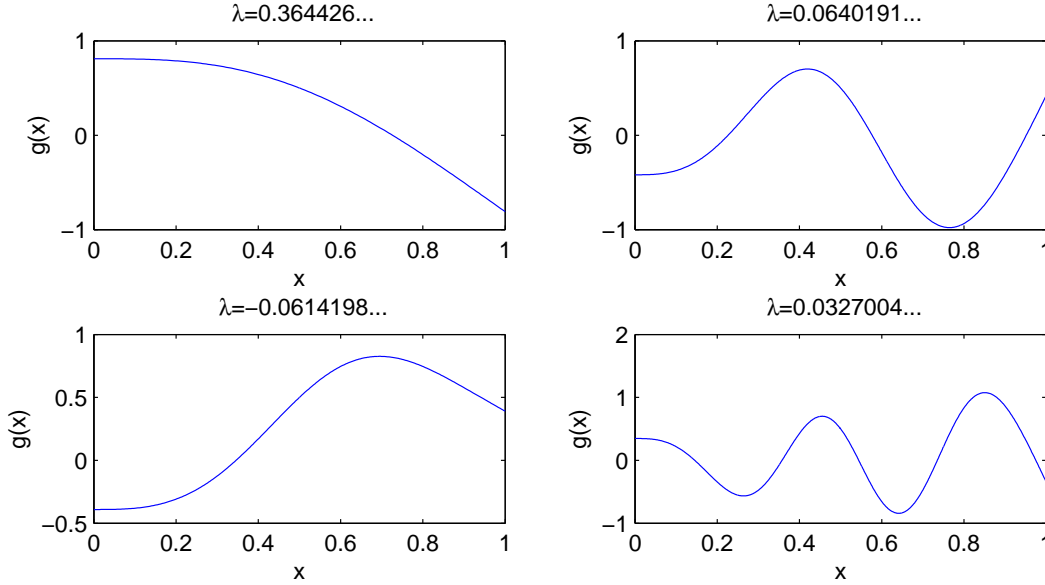


Figure 5: The eigenfunctions associated with the four largest (in absolute value) eigenvalues. The whole function $g(x)$, from 0 to 1 is found by using $g(x)$ between $1/2$ and 1 and $h(1-x)$ between 0 and $1/2$.

λ	$\langle g(x), \mathbf{1} \rangle$	$\ g(x)\ ^2$	c_n
0.364425573038	0.59039705924381	0.75905252379149	0.45921550437989
0.064019105418	-0.05366486422899	0.29041608589489	0.00991652250888
-0.06141983509	0.04799387821016	0.10956972701418	0.02102234265267
0.03270035262	0.02267022514715	0.24295945072711	0.00211532873771

Table 3: The values of λ , $\langle g(x), \mathbf{1} \rangle_U$, $\|g(x)\|_U^2$, and c_n for the first four eigenfunctions shown in Figure 5. The constant c_n is the ratio $\langle g(x), \mathbf{1} \rangle_U^2 / \|g(x)\|_U^2$ for the n^{th} eigenfunction.

The resulting predictions for the Euler numbers are shown in the last column of Table 1.

5 Concluding remarks

Another graph to investigate is the product with the even cycle C_{2m} , that is, $\mathcal{E}(G \square_S C_{2m})$. We conjecture that the resulting Euler number is asymptotically a constant times the associated Euler number for the product with a path, that is,

$$\frac{\mathcal{E}(G \square_S C_{2m})}{\mathcal{E}(G \square_S P_{2m})} \longrightarrow c,$$

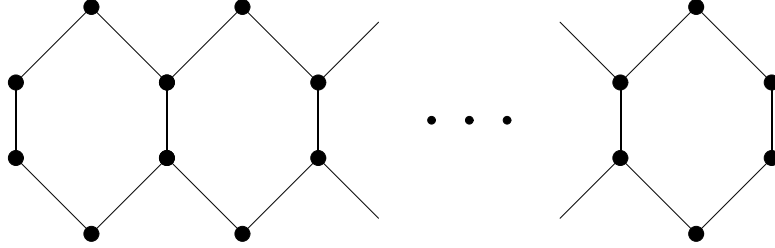


Figure 6: A bipartite graph H_m obtained by gluing m hexagons.

as m tends to infinity and c is a positive constant less than 1 for S non-empty.

Does the eigenfunction φ corresponding to the largest eigenvalue λ carry information about the distribution of entries in the first copy of G in an alternating labeling of $G \square_S P_m$? More specifically, in the case of alternating 2 by m arrays, let $\mathcal{E}(P_2 \square P_m; i, j)$ be the number of alternating arrays where the first column consists of the two entries i and j , where $1 \leq i < j \leq 2m$. Is the integer $\mathcal{E}(P_2 \square P_m; i, j)$ approximated by $c \cdot (2m)! \cdot \lambda^{m-1} \cdot (g(i/2m) + g(1 - j/2m))$ where c is the appropriate constant and g is the first eigenfunction displayed in Figure 5?

These techniques for obtaining the asymptotic behavior of the Euler numbers can be used for other classes of graphs as well. See for instance the graph H_m in Figure 6, which is built by gluing hexagons together. Although Theorem 1.1 does not directly apply to this class of graphs, one can extend the theory to obtain the same asymptotic result. Hence we have

$$\frac{\mathcal{E}(H_m)}{(4 \cdot m + 2)!} = c \cdot \lambda^{m-1} + O(\mu^{m-1}).$$

The essential question remaining is can the associated eigenvalue problem be solved explicitly.

Keeping n fixed we know that $\mathcal{E}(P_n \square P_m) \sim c_{(n)} \cdot (n \cdot m)! \cdot \lambda_{(n)}^{m-1}$ for a constant $c_{(n)}$ and the largest eigenvalue $\lambda_{(n)}$. Can anything be determined about the sequence $\lambda_{(n)}$? What can be said about the asymptotics of the Euler number $\mathcal{E}(P_m \square P_m)$ as m tends to infinity?

A different direction is to study the descent number of directed graphs (digraphs). For a digraph $G = (V, E)$ on n vertices define its *descent number* to be the number of labelings π of the vertices with 1 through n such that for each directed edge $u \rightarrow v$ we have that $\pi(u) < \pi(v)$. If the digraph contains a directed cycle then the descent number is zero. For an acyclic digraph (digraphs without directed cycles) the descent number is strictly positive. The classical descent set statistics for permutations is obtained by looking at orientations of the path. By gluing directed graphs together, one obtains classes of graphs whose asymptotics of the descent number is natural to study via linear operators and their eigenvalues.

The technique of translating a combinatorial problem into a problem of studying an operator and its spectrum was also explored in [1], where consecutive pattern avoiding in permutations were studied.

Finally, we end with a purely enumerative question for trees (connected graphs without cycles).

Conjecture 5.1. *For a tree T on n vertices the classical Euler number E_n is a lower bound for $\mathcal{E}(T)$,*

that is,

$$\mathcal{E}(T) \geq E_n.$$

Furthermore, equality only holds when the tree T is the path P_n .

Acknowledgments

The first author thanks Bob Strichartz and the Department of Mathematics at MIT where this paper was completed. The first author was partially supported by National Security Agency grant H98230-06-1-0072.

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